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BAYESIAN ESTIMATION OF SCALE PARAMETER OF INVERSE GAUSSIAN DISTRIBUTION USING LINES LOSS FUNCTION

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ABSTRACT

In this paper we have obtained Bayes estimator of the scale parameter of Inverse Gaussian Distribution. The loss function used is lines. A number of prior distributions have been considered and Bayes estimators have been compared with corresponding estimators with squared error loss function.

1. Introduction

The Inverse Gaussian (IG) is a long-tailed positively skewed distribution. Its shape is similar to that of the normal distribution. This distribution was originally derived as the first passage of lime distribution of Brownain motion with positive drift. In reliability and life-span applications it is primarily useful when there is substantial skewness. It has been studied extensively by Tweedie (1957 a, b) and Cohen and Whitten (1988). The Probability density function (pdf) of IG distribution is given by (Chhikara and Folks, 1989)

$$f(x;\ \mu,\theta) = \left(\frac{1}{2\pi\theta}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} e^{-\frac{(x-\mu)^2}{2\mu^2\theta x}}\ ; x > 0, \mu > 0, \theta > 0 \tag{1.1}$$

where θ is the scale parameter.

Suppose $\Delta = \frac{\hat{\theta}}{\theta} - 1$. Consider the following linex loss function

$$L(\Delta) = b\,[e^{a\Delta} - a\Delta - 1], a \neq 0, b > 0 \tag{1.2}$$

where $\hat{\theta}$ is an estimate of θ (Basu and Ebrahimi, 1991). Let us denote the posterior pdf of θ by $f\ (\theta\,/\,x)$, where x denotes a random sample x = (x_1,----, x_r). Let E_π stand for posterior expectation with respect to f $\left(\theta/\underline{x}\right)$. The posterior expectation of the loss function is given by equation (1.2) as

$$E_\pi[L(\Delta)] = b\left[e^a E_\pi\left(e^{a\hat{\theta}/\theta}\right) - a E_\pi\left(\frac{\hat{\theta}}{\theta} - 1\right) - 1\right] \quad (1.3)$$

The value of θ that minimizes (1.3) is denoted by $\hat{\theta}_A$ (Bayes estimator under L (Δ)) and may be obtained by solving the equation

$$\frac{d}{d\hat{\theta}} E_\pi[L(\Delta)] = 0$$

$$\Rightarrow E_\pi\left[\frac{1}{\theta}\left(e^{a\hat{\theta}/\theta}\right)\right] = e^a E_\pi\left[\frac{1}{\theta}\right] \quad (1.4)$$

if the expectations exist.

The objective of the present paper is to obtain a Bayes estimator of θ using a number of prior distributions. For the situation where we have no prior information about the parameter θ, we may use the quasi-density

$$g_1(\theta) = \frac{1}{\theta^d}\,; \quad \theta > 0, d \geq 0 \quad (1.5)$$

here d = 0 leads to a diffuse prior and d = 1, a non-informative prior.

The most widely used prior distribution of θ is the inverted gamma with parameters $\alpha, \beta > 0$, with density function given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\alpha}\left(\frac{1}{\theta}\right)^{\alpha+1} e^{-\beta/\theta} & ; \theta > 0 \\ 0 & ; \text{otherwise} \end{cases} \quad (1.6)$$

if it is known in advance that the probable value of θ lies over a finite range $[\alpha, \beta]$, but do not have any strong opinion about any subset of values this range. In such a case a uniform distribution over $[\alpha, \beta]$ may be appropriate.

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha} & ;\ 0 < \alpha \leq \theta \leq \beta \\ 0 & ;\ \text{otherwise} \end{cases} \quad (1.7)$$

2. Bayes Estimators under g_1 (θ)

Let us suppose that n items are put to life test and terminate the experiment when r ($\leq n$) items have failed. If x_1,-------------, x_r denote the first r observations, then the joint pdf is given by

$$f(\underline{x} : \theta) = \frac{(n)!}{(n-r)!} (Z\pi\theta)^{-r/2} \left(\prod_{i=1}^{r} X_i^{-3/2} \right) . e^{-Z/\theta} \tag{2.1}$$

where , $Z = \frac{1}{2\,\mu^2} \left(\sum_{i=1}^{r} \frac{(X_i - \mu)^2}{X_i} + (n-r) \frac{(X_r - \mu)^2}{X_r} \right)$

The maximum likelihood estimator (MLE) of θ is given by

$$\hat{\theta} = \frac{2Z}{r} \tag{2.2}$$

The posterior pdf of θ is obtained as

$$f(\theta/\underline{x}) = \frac{Z^{\left(\frac{r}{2}+d-1\right)}}{\Gamma\left(\frac{r}{2}+d-1\right)} \theta^{-\left(\frac{r}{2}+d\right)} e^{-Z/\theta} \tag{2.3}$$

Thus, using (1.2), the Bays estimator of θ relative to L (Δ) comes out to be

$$\hat{\theta}_A = \frac{1}{a}\left[1 - e^{-a\left(\frac{r}{2}+d\right)}\right] Z \tag{2.4}$$

Here a is positive, since we assume that overestimating θ is more costly than under estimating it. The Bayes estimator of θ under squared error loss function (SELF) is given by

$$\hat{\theta}_s = \left[\frac{Z}{\frac{r}{2}+d-2}\right] \tag{2.5}$$

The risk function of estimators $\hat{\theta}_A$ and $\hat{\theta}_S$ relative to L (Δ) are denoted by R_A ($\hat{\theta}_A$) and R_A ($\hat{\theta}_S$) respectively, and are as follows

$$R_A(\hat{\theta}_A) = b\left[e^{-ad\left(\frac{r}{2}+d\right)} - \frac{r}{2}\left(1 - e^{-a\left(\frac{r}{2}+d\right)} + a - 1\right)\right] \tag{2.6}$$

$$R_A(\hat{\theta}_S) = b\left[\left(1 - \frac{a}{\frac{r}{2} + d - 2}\right)^{-\frac{r}{2}} - \left(\frac{\frac{ar}{2}}{\frac{r}{2} + d - 2}\right) + a - 1\right] \tag{2.7}$$

The risk functions of the estimators $\hat{\theta}_A$ and $\hat{\theta}_S$ relative to squared error loss $\left(\theta - \hat{\theta}\right)^2$ are denoted by $R_S\left(\hat{\theta}_A\right)$ and $R_S\left(\hat{\theta}_S\right)$ respectively and we have

$$R_S\left(\hat{\theta}_A\right) = \theta^2\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)}{a^2}\left\{1 - e^{-a}\left(\frac{r}{2} + d\right)\right\}^2\right.$$

$$\left. - \frac{r}{a}\left\{1 - e^{-a6}\left(\frac{r}{2} + d\right) + 1\right\}\right] \tag{2.8}$$

$$R_S\left(\hat{\theta}_S\right) = \theta^2\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)}{\left(\frac{r}{2}+d-2\right)^2} - \frac{r}{\left(\frac{r}{2}+d-2\right)} + 1\right] \tag{2.9}$$

In general, neither of the estimators uniformly dominates the other. For example, if a = 1, r = 10, d = 1, then

$$\frac{R_S\left(\hat{\theta}_S\right)}{\theta^2} = 0.375 > 0.1719\ \frac{R_S\left(\hat{\theta}_A\right)}{\theta^2}$$

and

$$\frac{R_A\left(\hat{\theta}_S\right)}{b} = 0.3002 > 0.0789\ \frac{R_A\left(\hat{\theta}_A\right)}{b}$$

If a = 1, r = 10, d = 3, then

$$\frac{R_S\left(\hat{\theta}_S\right)}{\theta^2} = 0.1667 < 0.2392\ \frac{R_S\left(\hat{\theta}_A\right)}{\theta^2}$$

and

$$\frac{R_A\left(\hat{\theta}_S\right)}{b} = 0.0821 < 0.0998\ \frac{R_A\left(\hat{\theta}_A\right)}{b}$$

3. Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, using (2.1), we obtain the posterior distribution as

$$f\left(\theta/\underline{x}\right) = \frac{(\beta+Z)^{\frac{r}{2}+\alpha}}{\left|\overline{\frac{r}{2}+\alpha}\right.} \theta^{-\left(\frac{r}{2}+\alpha+1\right)} e^{-(\beta+Z)/\theta} \quad ; \theta > 0 \tag{3.1}$$

Thus, using (1.2), the Bayes estimator of θ relative to L (Δ) is obtained as

$$\hat{\theta}_A = \frac{1}{\alpha}\left[1 - e^{-a\left(\frac{r}{2}+\alpha+1\right)}\right](\beta+Z) \tag{3.2}$$

The Bayes estimator of θ under SELF is given by

$$\hat{\theta}_S = \left(\frac{\beta+Z}{\frac{r}{2}+\alpha-1}\right) \tag{3.3}$$

The risk functions of the estimators $\hat{\theta}_A$ and $\hat{\theta}_S$ relative to L $(\Delta)$are then given by

$$R_A(\hat{\theta}_A) = b\left[\left[\exp\left\{-a(\alpha+1)\left(\frac{r}{2}+\alpha+1\right)\right\}\right]\right.$$

$$\left[\exp\left\{\frac{\beta}{\theta}\left[1-\exp\left\{-a\left(\frac{r}{2}+\alpha+\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.1\right)\right\}\right]\right\}\right] - \left[1-\exp\left\{-a\left(\frac{r}{2}+\alpha+1\right)\right\}\right]\left(\frac{r}{2}+\frac{\beta}{\theta}\right) + a - 1\right] \tag{3.4}$$

and,

$$R_A(\hat{\theta}_S) = b\left[\left[\exp\left\{-a\left[\frac{(1-\beta)}{\theta\left(\frac{r}{2}+\alpha+1\right)}\right]\right\}\right]\right.$$

$$\left.\left[1 - \frac{a}{\left(\frac{r}{2}+\alpha-1\right)}\right]^{-\frac{r}{2}} - \left[\frac{a\left(\frac{r}{2}+\frac{\beta}{\theta}\right)}{\left(\frac{r}{2}+\alpha-1\right)}\right] + a - 1\right] \tag{3.5}$$

The Bayes risks for the estimators $\left(\hat{\theta}_A\right)$ and $\left(\hat{\theta}_S\right)$ are denoted by $r_A\left(\hat{\theta}_A\right)$ and $r_A\left(\hat{\theta}_S\right)$ respectively, and are given by

$$r_A\left(\hat{\theta}_A\right) = b\left[a - \left(\frac{r}{2} + \alpha + 1\right)\left\{1 - \exp\left(-a\left(\frac{r}{2} + \alpha + 1\right)\right)\right\}\right] \quad (3.6)$$

and

$$r_A\left(\hat{\theta}_S\right) = b\left[e^{-a}\left[1 - \frac{a}{\left(\frac{r}{2}+\alpha-1\right)}\right]^{-\left(\frac{r}{2}+\alpha\right)} - \left[1 + \frac{a}{\left(\frac{r}{2}+\alpha-1\right)}\right]\right] \quad (3.7)$$

The risk functions and Bayes risks of $\left(\hat{\theta}_S\right)$ and $\left(\hat{\theta}_A\right)$ under SELF are similarly obtained and are given by

$$R_S\left(\hat{\theta}_A\right) = \theta^2\left[\left[\frac{1-\exp\left\{-a\left(\frac{r}{2}+\alpha+1\right)\right\}}{a}\right]^2\left[\frac{r}{2}\left(\frac{r}{2}+1\right)\right.\right.$$

$$\left. + \frac{r\beta}{\theta} + \frac{\beta^2}{\theta^2}\right] - \frac{2}{a}\left[1 - \exp\left\{-a\left(\frac{r}{2} + \right.\right.\right.$$

$$\left.\left.\left.\alpha + 1\right)\right\}\right]\left[\frac{r}{2} + \frac{\beta}{\theta}\right] + 1\Bigg] \quad (3.8)$$

$$R_S\left(\hat{\theta}_S\right) = \theta^2\left[\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)+\frac{r\beta}{\theta}+\frac{\beta^2}{\theta^2}}{\left(\frac{r}{2}+\alpha-1\right)^2}\right] - \frac{2\left[\frac{r}{2}+\frac{\beta}{\theta}\right]}{\left(\frac{r}{2}+\alpha-1\right)} + 1\right] \quad (3.9)$$

$$r_S\left(\hat{\theta}_A\right) = \beta^2\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)K^2 - rK + 1}{(\alpha-1)(\alpha-2)} + \frac{2K(rK/2-1)}{(\alpha-1)} + K^2\right] \quad (3.10)$$

where,

$$K = \frac{1}{a}\left[1 - \exp\left\{-a\left(\frac{r}{2} + \alpha + 1\right)\right\}\right]$$

and,

$$r_S(\hat{\theta}_S) = \frac{\beta^2}{(\alpha-1)(a-2)\left(\frac{r}{2}+a-1\right)} \tag{3.11}$$

4. Bayes Estimators under $g_3(\theta)$

Under $g_3(\theta)$, using (2.1), the posterior distribution is given by

$$f(\theta/\underline{x}) = \frac{Z^{\frac{r}{2}-1}\,\theta^{-\frac{r}{2}}\,e^{-z/\theta}}{Ig\left[\frac{Z}{\alpha},\frac{r}{2}-1\right]-Ig\left[\frac{Z}{\beta},\frac{r}{2}-1\right]} \qquad ; 0<\alpha \le \theta \le \beta \tag{4.1}$$

Where $Ig(x,n) = \int_0^x e^{-t}\, t^{n-1}\, dt$ is the incomplete gamma function.

Using (1.2), the Bayes estimator of θ relative to L (Δ) is $\hat{\theta}_A$, where $\hat{\theta}_A$ is the solution of the following equation

$$e^{-a}\left[\frac{Ig\left(\frac{Z}{\alpha},\frac{r}{2}\right)-Ig\left(\frac{Z}{\beta},\frac{r}{2}\right)}{Ig\left(\frac{Z-a\,\hat{\theta}_A}{\alpha},\frac{r}{2}\right)-Ig\left(\frac{Z-a\,\hat{\theta}_A}{\beta},\frac{r}{2}\right)}\right] = \left[\frac{Z}{Z-a\,\hat{\theta}_A}\right]^{\frac{r}{2}} \tag{4.2}$$

The equation (4.2) can be solved numerically. The Bayes estimator of θ under SELF is given by

$$\hat{\theta}_S = \left[\frac{Ig\left(\frac{Z}{\alpha},\frac{r}{2}-2\right)-Ig\left(\frac{Z}{\beta},\frac{r}{2}-2\right)}{Ig\left(\frac{Z}{\alpha},\frac{r}{2}-1\right)-Ig\left(\frac{Z}{\beta},\frac{r}{2}-1\right)}\right]Z \tag{4.3}$$

In this case risk functions and Bayes risks cannot be obtained in a closed form.

Conclusion

From the given example in section (2), it is clear that neither of the estimators uniformly dominates the other. We therefore recommend that the estimator's choice lies according to the value of d in the quasi density used as the prior distribution which in turn depends on the situation at hand.

Both risk functions, given by (3.4) and (3.5) under L (Δ), depend on θ and neither of the estimators uniformly dominates the other. Risk functions given by

(3.8) and (3.9) are also depend on θ and neither of these two estimators uniformly dominates the other under squared error loss.

It is clear from the equations (4.2) and (4.3) that only numerical solution exists for the estimators $\hat{\theta}_A$ and $\hat{\theta}_S$. In this case, the risk functions and Bayes risks cannot be obtained in closed forms. Thus, the comparison could only be done after obtaining the results numerically, which depends on the value of the parameter itself.

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